Radius of Convergence for Almost Leinert Sets

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Introduction

It has been established that random paths on mathematical trees with a number *n* generators can be expressed using matrices.

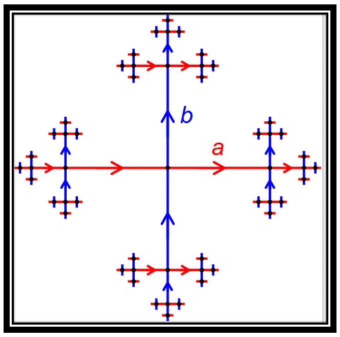
These paths can either return to the origin or diverge with no way back. This divergence behavior can be understood by studying the radius of convergence.

Background

- The set of all possible words built from a set *S*, form an algebraic group called the free group.
- Leinert Set: Set of generators of free group followed by their inverses which do not reduce to the identity.
- Bad strings: words that can be reduced to "1".
- Almost Leinert: Set followed by inverses with no restrictions of bad strings.

Numerical Algorithms

The techniques used to experimentally verify the analytical results in this paper was to build an algorithm to experimentally reduce the number of bad strings using two key techniques: parity reduction and normal form. The string can only reduce if all the generators can be adjacent to its inverse. If the string was not reducible, we added it into our calculations. Finally, evaluated each function and calculated it relative error.

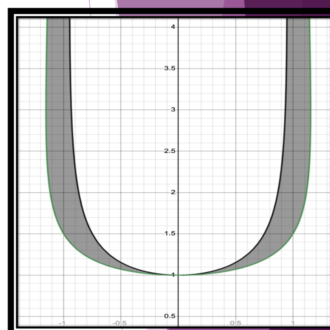


Results

When studying the radius of convergence of product of free groups, a specific instance of almost Leinert set. We found a function *Q(z, G(z))* that incorporates the number of bad strings in the set such that,

$$P(zG(z)) \leq G(z) \leq Q(z, G(z)).$$

In our numerical calculations with increasing dimensions in the group formed by the product of two free groups with two generators, the lower bound on radius of convergence for *G(z)* differed from the upper bound by ~3% relative error.



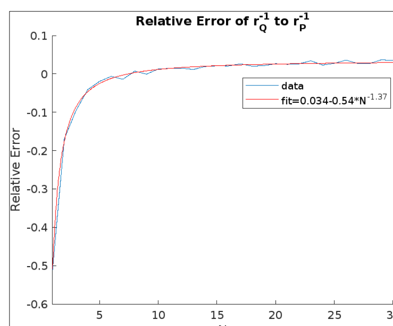
$$P(t) = 1 + \frac{1}{2}\sum_{x \in X}\left(\sqrt{1 + 4|\alpha(x)|^2 t^2} - 1\right)$$

$$Q(t, z) = 1 + \frac{1}{2}\sum_{i,j}\left(\sqrt{\left(1 + \frac{zt^2}{c^2 - t^2}\right)^2 + 4\alpha_{i,j}^2 z^2 t^2} + \frac{zt^2}{c^2 - t^2} - 1\right)$$

Summary

Although our results are still inconclusive, results seem to be following our intuition created by past studies. This new function has been successful in tightly bounding the radius of convergence of almost Leinert Sets with the radius of convergence found by Woess for Leinert Sets.

Further calculations and analysis must be conducted to further our current understanding over the topic.



Future Directions

Future works should consider analytically exploring a tighter bound on the "irreducible" strings as the size of the string exponentially decrease in proportion to valid strings. For the purposes of this project, only bad strings up to length sixteen were found, and the minimum length of the string found was length 8. As the length increases and bad strings are added in, the number of bad strings are seen to exponentially increase in proportion to valid strings.

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